

NEW TYPE INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS

KHALED MEHREZ AND PRAVEEN AGARWAL

ABSTRACT. We have recently established some integral inequalities for convex functions via the Hermite–Hadamard’s inequalities. In continuation here, we also establish some interesting new integral inequalities for convex functions via the Hermite–Hadamard’s inequalities and Jensen’s integral inequality. Useful applications involving special means are also included.

Keywords: Hermite–Hadamard inequality, Integral inequalities, Convex functions, Special means.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, $a, b \in I$ with $a < b$, if and only if,

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

is well known in the literature as the Hermite–Hadamard inequality for convex function. A vast literature related to (1.1) have been produced by a large number of mathematicians [7] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [5], Information Theory [3], Operator Theory [6] and others.

Very recently, authors [9] established some new type integral inequalities for convex function via the Hermite–Hadamard inequality. This paper is a continuation of some line of authors results in [9]. Motivated by above work here, we proved some interesting new type integral inequalities for differentiable convex functions by using the Hermite–Hadamard inequality and Jensen integral inequality. As applications, we obtain some new inequality involving special means of real numbers.

In the proof of the main results we will need the following two lemmas.

Lemma 1. [2] *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 , and $a, b \in I$ with $a < b$, then we have*

$$(1.2) \quad \frac{f(b)+f(a)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

Lemma 2. [8] *(Jensen inequality) Let μ be a probability measure and let $\varphi \geq 0$ be a convex function. Then, for all f be a integrable function we have*

$$(1.3) \quad \int \varphi \circ f d\nu \geq \varphi\left(\int f d\nu\right).$$

2. MAIN RESULTS

Now we are ready to present our main results asserted by Theorems 1 to 7.

Theorem 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $0 < a < b$. If $|f'|$ is convex and increasing on $[a, b]$, then the following inequality

$$(2.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} \right| \leq \frac{|a||f'(a)| + |b||f'(b)|}{2}.$$

Proof. Using integration by parts, which is verified under the conditions given in the theorem, we have

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{bf(b) - af(a)}{b-a} \right| \leq \left(\frac{1}{b-a} \int_a^b |t||f'(t)| dt \right).$$

On the other hand, using the fact that the functions $|f'(t)|$ is convex and increasing on $[a, b]$ and the $|t|$ is convex and increasing on $[a, b]$, $a, b > 0$, thus the function $|t||f'(t)|$ is also convex on $[a, b]$, as product of positive convex and increasing functions. Now, by the right hand side inequality (1.1) we deduce that inequality (2.2) is valid. ■

Theorem 2. Let $p > 1, q \geq 1$ and $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $0 < a < b$. If $|f'|^q$ is convex and increasing, then the following inequality

$$(2.3) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} \right| \leq \frac{(|f'(b)| + |f'(a)|)^{\frac{p-1}{p}} (|b|^p + |a|^p)^{\frac{q-1}{qp}} \cdot (|b|^p |f'(b)|^q + |a|^p |f'(a)|^q)^{\frac{1}{qp}}}{2^{2-1/p}}.$$

Proof. Applying the $p > 1$ on the inequality (2.2), we have

$$(2.4) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{bf(b) - af(a)}{b-a} \right|^p \leq \left(\frac{1}{b-a} \int_a^b |t||f'(t)| dt \right)^p.$$

Now, we set $\varphi(t) = |t|^p$, $f(t) = t$ and $d\mu(t) = \frac{|f'(t)| dt}{\int_a^b |f'(t)| dt}$. So, by means of Lemma 2, we get

$$(2.5) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{bf(b) - af(a)}{b-a} \right|^p \leq \left(\frac{\int_a^b |f'(t)| dt}{b-a} \right)^{p-1} \cdot \left(\frac{1}{b-a} \int_a^b |t|^p |f'(t)| dt \right)$$

By the power-mean inequality, we have

$$(2.6) \quad \frac{1}{b-a} \int_a^b |t|^p |f'(t)| dt \leq \frac{1}{b-a} \left(\int_a^b |t|^p dt \right)^{1-\frac{1}{q}} \cdot \left(\int_a^b |t|^p |f'(t)|^q dt \right)^{\frac{1}{q}}$$

Since the functions $|f'(t)|^q$ is convex on $[a, b]$ and the $|t|^p$ is convex on $[a, b]$, for each $p > 1$, thus the function $|t|^p |f'(t)|^q$ is also convex on $[a, b]$, as product of positive convex functions. By again of the right hand side inequality (1.1), we have

$$(2.7) \quad \frac{1}{b-a} \int_a^b |t|^p |f'(t)|^q dt \leq \frac{|b|^p |f'(b)|^q + |a|^p |f'(a)|^q}{2}$$

and

$$(2.8) \quad \frac{1}{b-a} \int_a^b |t|^p dt \leq \frac{|b|^p + |a|^p}{2}.$$

According to (2.6), (2.7) and (2.8), we have

$$(2.9) \quad \frac{1}{b-a} \int_a^b |t|^p |f'(t)| dt \leq \frac{(|b|^p + |a|^p)^{1-\frac{1}{q}} \cdot (|b|^p |f'(b)|^q + |a|^p |f'(a)|^q)^{\frac{1}{q}}}{2}.$$

Again, from the right hand side of inequality (1.1), we have

$$(2.10) \quad \frac{1}{b-a} \int_a^b |f'(t)| dt \leq \frac{|f'(b)| + |f'(a)|}{2}.$$

In view of (2.5), (2.10) and (2.9) we obtain the desired result. ■

Theorem 3. Let $p > 1, q \geq 1$ and $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $0 < a < b$. If $|f'|^q$ is convex and increasing on $[a, b]$, then the following inequality

$$(2.11) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} \right| \leq \frac{(|f'(b)| + |f'(a)|)^{\frac{p-1}{p}} (b^{p+1} - a^{p+1})^{\frac{q-1}{qp}} \cdot (|b|^p |f'(b)|^q + |a|^p |f'(a)|^q)^{\frac{1}{q}}}{2^{2-\frac{1}{p}} (p+1)^{\frac{q-1}{qp}}}.$$

Proof. The proof is parallel to that of Theorem 2 by replacing equation (2.8) by

$$\int_a^b |t|^p dt = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

We omit the further details. ■

Remark 1. Suppose that all the assumptions of Theorem 3 are satisfied with $|f'| \leq M$, and $0 \leq a < b$. we get

$$(2.12) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} \right| \leq \frac{M^{\frac{p-1+pq}{p}} (b^{p+1} - a^{p+1})}{2^{1/p} (b-a)}$$

where $p > 1, q \geq 1$.

Here, by using the classical definitions of Beta function and gamma function, we establish certain interesting and new inequalities are given by the next Theorems. For our purpose, We recall the Beta function and gamma function defined by (see [10])

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad \text{and} \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

The Beta function satisfied the following properties:

$$B(x, x) = 2^{1-2x} B(1/2, x), \quad \text{and} \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

In particular, we have

$$B(p+1, p+1) = 2^{1-2(p+1)} B(1/2, p+1) = 2^{1-2(p+1)} \frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} = 2^{1-2(p+1)} \frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+3/2)}.$$

Theorem 4. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and suppose that f has 3 derivatives f', f'' and f''' on I^0 with $a < b$. If $|f''|^q$ is convex on $[a, b]$ and $|f'''|^q$ is convex and increasing on $[a, b]$, then the following

inequality

$$(2.13) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} - \frac{bf'(b) + af'(a)}{2} \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{\frac{q-1}{q}} \left\{ \left[\frac{|2f''(a)|^q + |2f''(b)|^q}{12} \right]^{\frac{1}{q}} + \left[\frac{|af'''(a)|^q + |bf'''(b)|^q}{12} \right]^{\frac{1}{q}} \right\}.$$

Proof. From Lemma 1, we get

$$(2.14) \quad \frac{1}{b-a} \int_a^b x f'(x) dx = \frac{af'(a) + bf'(b)}{2} - \frac{(b-a)^2}{2} \left[2 \int_0^1 t(1-t) f^{(2)}(ta + (1-t)b) dt + \int_0^1 t(1-t) F(ta + (1-t)b) dt \right],$$

where $F(t) = t f^{(3)}(t)$. From the Hölder inequality, we obtain

$$(2.15) \quad \begin{aligned} \int_0^1 t(1-t) |f^{(2)}(ta + (1-t)b)| dt &= \int_0^1 [t(1-t)]^{1-\frac{1}{q}} [t(1-t)]^{\frac{1}{q}} |f''(ta + (1-t)b)| dt \\ &\leq \left[\int_0^1 t(1-t) dt \right]^{\frac{q-1}{q}} \left[\int_0^1 t(1-t) |f''(ta + (1-t)b)|^q dt \right]^{\frac{1}{q}} \\ &\leq \left[\int_0^1 t(1-t) dt \right]^{\frac{q-1}{q}} \left[|f''(a)|^q \int_0^1 t^2(1-t) dt + |f''(b)|^q \int_0^1 t(1-t)^2 dt \right]^{\frac{1}{q}} \\ &= \left(\frac{1}{6} \right)^{\frac{q-1}{q}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{12} \right]^{\frac{1}{q}} \end{aligned}$$

Since the function $|f'''(t)|^q$ is convex then the function $|tf'''(t)|^q$ is convex as a product of two positive convex and increasing functions. So, for every $t \in [0, 1]$ we have

$$(2.16) \quad |F(ta + (1-t)b)|^q \leq t|F(a)|^q + (1-t)|F(b)|^q = t|af'''(a)|^q + (1-t)|bf'''(b)|^q.$$

Hence, from (2.16) and the Hölder inequality we have

$$(2.17) \quad \begin{aligned} \int_0^1 t(1-t) |F(ta + (1-t)b)| dt &= \int_0^1 [t(1-t)]^{1-\frac{1}{q}} [t(1-t)]^{\frac{1}{q}} |F(ta + (1-t)b)| dt \\ &\leq \left[\int_0^1 t(1-t) dt \right]^{\frac{q-1}{q}} \left[\int_0^1 t(1-t) |F(ta + (1-t)b)|^q dt \right]^{\frac{1}{q}} \\ &\leq \left[\int_0^1 t(1-t) dt \right]^{\frac{q-1}{q}} \left[|F(a)|^q \int_0^1 t^2(1-t) dt + |F(b)|^q \int_0^1 t(1-t)^2 dt \right]^{\frac{1}{q}} \\ &= \left(\frac{1}{6} \right)^{\frac{q-1}{q}} \left[\frac{|af'''(a)|^q + |bf'''(b)|^q}{12} \right]^{\frac{1}{q}}. \end{aligned}$$

So, the proof of Theorem 4 is completes. ■

Another similar result is embodied in the following theorem.

Theorem 5. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and suppose that f has 3 derivatives f' , f'' and f''' on I^0 with $a < b$. If $|f''|^q$ is convex on $[a, b]$ and $|f'''|^q$ is convex and increasing on $[a, b]$, then the following

inequality

$$(2.18) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} - \frac{bf'(b) + af'(a)}{2} \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\frac{1}{(q+1)(q+2)(q+3)} \right]^{\frac{1}{q}} \left\{ \left[2|2f''(a)|^q + (q+1) \right. \right. \\ \left. \left. \times |2f''(b)|^q \right]^{\frac{1}{q}} + \left[2|af'''(a)|^q + (q+1)|bf'''(b)|^q \right]^{\frac{1}{q}} \right\},$$

holds for all $q \geq 1$.

Proof. Using the power-mean inequality, we have

$$(2.19) \quad \int_0^1 t(1-t)|f''(ta + (1-t)b)| dt \leq \left[\int_0^1 t dt \right]^{1-\frac{1}{q}} \left[\int_0^1 t(1-t)^q |f''(ta + (1-t)b)|^q dt \right]^{\frac{1}{q}} \\ \leq \left[\int_0^1 t dt \right]^{1-\frac{1}{q}} \left[|f''(a)|^q \int_0^1 t^2(1-t)^q dt + |f''(b)|^q \int_0^1 t(1-t)^{q+1} dt \right]^{\frac{1}{q}} \\ = \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\frac{2}{(q+1)(q+2)(q+3)} |f''(a)|^q + \frac{1}{(q+2)(q+3)} |f''(b)|^q \right]^{\frac{1}{q}} \\ = \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\frac{1}{(q+1)(q+2)(q+3)} \right]^{\frac{1}{q}} \left[2|f''(a)|^q + (q+1)|f''(b)|^q \right]^{\frac{1}{q}}.$$

In the same way, we get

$$(2.20) \quad \int_0^1 t(1-t)|F(ta + (1-t)b)| dt \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\frac{1}{(q+1)(q+2)(q+3)} \right]^{\frac{1}{q}} \left[2|af'''(a)|^q + (q+1)|bf'''(b)|^q \right]^{\frac{1}{q}}.$$

Combining (2.14), (2.19) and (2.20) we deduce that the inequality (2.18) holds. ■

Theorem 6. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and suppose that f has 3 derivatives f', f'' and f''' on I^0 with $a < b$. If $|f''|^q$ is convex on $[a, b]$ and $|f'''|^q$ is convex and increasing on $[a, b]$, then the following inequality

$$(2.21) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} - \frac{bf'(b) + af'(a)}{2} \right| \leq \frac{(b-a)^2}{2} \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2^{1+2p}\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left\{ \left[\frac{|2f''(a)|^q + |2f''(b)|^q}{2} \right]^{\frac{1}{q}} \right. \\ \left. + \left[\frac{|af'''(a)|^q + |bf'''(b)|^q}{2} \right]^{\frac{1}{q}} \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Again from the Hölder inequality, we have

$$\begin{aligned}
 (2.22) \quad \int_0^1 t(1-t)|f''(ta+(1-t)b)|dt &\leq \left[\int_0^1 t^p(1-t)^p dt \right]^{\frac{1}{p}} \left[\int_0^1 |f''(ta+(1-t)b)|^q dt \right]^{\frac{1}{q}} \\
 &\leq \left[\int_0^1 t^p(1-t)^p dt \right]^{\frac{1}{p}} \left[|f''(a)|^q \int_0^1 t dt + |f''(b)|^q \int_0^1 (1-t) dt \right]^{\frac{1}{q}} \\
 &= \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2^{1+2p}\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}.
 \end{aligned}$$

In the same way we obtain

$$\begin{aligned}
 (2.23) \quad \int_0^1 t(1-t)|F'(ta+(1-t)b)|dt &\leq \left[\int_0^1 t^p(1-t)^p dt \right]^{\frac{1}{p}} \left[\int_0^1 |F'(ta+(1-t)b)|^q dt \right]^{\frac{1}{q}} \\
 &\leq \left[\int_0^1 t^p(1-t)^p dt \right]^{\frac{1}{p}} \left[|F'(a)|^q \int_0^1 t dt + |F'(b)|^q \int_0^1 (1-t) dt \right]^{\frac{1}{q}} \\
 &= \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2^{1+2p}\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left[\frac{|af'''(a)|^q + |bf'''(b)|^q}{2} \right]^{\frac{1}{q}}.
 \end{aligned}$$

In view of (2.25) and (2.23) we deduce that the inequality (2.24) holds true. \blacksquare

Theorem 7. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and suppose that f has 3 derivatives f' , f'' and f''' on I^0 with $a < b$. If $|f''|^q$ is convex and $|f'''|^q$ is convex and increasing on $[a, b]$, then the following inequality

$$\begin{aligned}
 (2.24) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{bf(b) - af(a)}{b-a} - \frac{bf'(b) + af'(a)}{2} \right| &\leq \frac{(b-a)^2}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left[\frac{|2f''(a)|^q + (q+1)|2f''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right. \\
 &\quad \left. + \left[\frac{|af'''(a)|^q + (q+1)|bf'''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right\},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using the Hölder inequality

$$\begin{aligned}
 (2.25) \quad \int_0^1 t(1-t)|f''(ta+(1-t)b)|dt &\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[\int_0^1 (1-t)^q |f''(ta+(1-t)b)|^q dt \right]^{\frac{1}{q}} \\
 &\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[|f''(a)|^q \int_0^1 t(1-t)^q dt + |f''(b)|^q \int_0^1 (1-t)^{q+1} dt \right]^{\frac{1}{q}} \\
 &= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{|f''(a)|^q + (q+1)|f''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}}.
 \end{aligned}$$

On the other hand, we get

(2.26)

$$\begin{aligned} \int_0^1 t(1-t)|F(ta+(1-t)b)|dt &\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[\int_0^1 (1-t)^q |F(ta+(1-t)b)|^q dt \right]^{\frac{1}{q}} \\ &\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[|F(a)|^q \int_0^1 t(1-t)^q dt + |F(b)|^q \int_0^1 (1-t)^{q+1} dt \right]^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{|af'''(a)|^q + (q+1)|bf'''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. ■

3. APPLICATIONS

In this section, we shall use the results of Section 2 to prove by simple computation the following new inequalities connecting the above means for arbitrary real numbers.

1. The arithmetic mean:

$$A = A(a, b) = \frac{a+b}{2}; \quad a, b \in \mathbb{R}.$$

2. The generalized logarithmic mean:

$$L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \right]^{\frac{1}{n}}; \quad n \in \mathbb{N}, n \geq 1, \quad a, b \in \mathbb{R}.$$

Proposition 1. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$ such that $n \geq 1$. Then the following inequality*

$$(3.27) \quad L_n^n(a, b) \leq A(|a|^n, |b|^n)$$

Proof. The proof is immediate from Theorem 1 where $f(x) = x^n$, $n \geq 1$. ■

Remark 2. *We note that the inequality (3.27) is not new, was proved by Agarwal and Dragomir in [1].*

Proposition 2. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$ such that $n \geq 3$. Then the following inequality*

(3.28)

$$|L_n^n(a, b)| \leq 2n^{\frac{1-q}{pq}} [A(|a|^{n-1}, |b|^{n-1})]^{\frac{p-1}{p}} [A(|a|^n, |b|^n)]^{\frac{q-1}{qp}} \left[A(|a|^{n(p+q)-q}, |b|^{n(p+q)-q}) \right]^{\frac{1}{pq}}$$

holds for all $p > 1$ and $q \geq 1$.

Proof. The proof is immediate from Theorem 2 with $f(x) = x^n$, $x \in \mathbb{R}$, $n \geq 3$. ■

Proposition 3. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$ such that $n \geq 3$. Then the following inequalities*

$$(3.29) \quad |L_n^n(a, b) + A(a^n, b^n)| \leq \min \{ K_1^{(n,q)}(a, b), K_2^{(n,q)}(a, b) \} \frac{n(n-1)(b-a)^2}{4}$$

and

$$(3.30) \quad |L_n^n(a, b)| \leq \min \{ K_1^{(n,q)}(a, b), K_2^{(n,q)}(a, b) \} \frac{n(n-1)(b-a)^2}{8}, \quad a, b > 0,$$

holds true for all $q \geq 1$, where

$$K_1^{(n,q)}(a, b) = \frac{1}{3} \left[A \left(|a|^{(n-2)q}, |b|^{(n-2)q} \right) \right]^{\frac{1}{q}}$$

$$K_2^{(n,q)}(a, b) = \left[\frac{4}{(q+1)(q+2)(q+3)} \right]^{\frac{1}{q}} \left[A \left(2|a|^{(n-2)q}, (q+1)|b|^{(n-2)q} \right) \right]^{\frac{1}{q}}$$

Proof. From Theorem 4 and Theorem 5 for $f(x) = x^n$, we obtain (3.29). Finally, combining (3.29) and (3.27) we deduce that the inequality (3.30) holds true. ■

Remark 3. We note that if $q = 1$, we have $K_1^{(n,1)}(a, b) = K_2^{(n,1)}(a, b)$, for all $a, b \in \mathbb{R}$ and $n \geq 3$. Consequently, we obtain that

$$(3.31) \quad |L_n^n(a, b)| \leq \frac{n(n-1)(b-a)^2}{24} A \left(|a|^{(n-2)}, |b|^{(n-2)} \right).$$

Proposition 4. Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$ such that $n \geq 3$. Then the following inequality

$$(3.32) \quad |L_n^n(a, b) + A(a^n, b^n)| \leq \min \left\{ K_3^{(n,p,q)}(a, b), K_4^{(n,p,q)}(a, b) \right\} \frac{n(n-1)(b-a)^2}{2}$$

and

$$(3.33) \quad |L_n^n(a, b)| \leq \min \left\{ K_3^{(n,p,q)}(a, b), K_4^{(n,p,q)}(a, b) \right\} \frac{n(n-1)(b-a)^2}{4}, \quad a, b > 0,$$

holds true for all $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, where

$$K_3^{(n,p,q)}(a, b) = \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2^{1+2p}\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left[A \left(|a|^{(n-2)q}, |b|^{(n-2)q} \right) \right]^{\frac{1}{q}}$$

$$K_4^{(n,p,q)}(a, b) = \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{2}{(q+1)(q+2)} \right]^{\frac{1}{q}} \left[A \left(|a|^{(n-2)q}, (q+1)|b|^{(n-2)q} \right) \right]^{\frac{1}{q}}$$

Proof. The inequality (3.32) follows from Theorem 6 and Theorem 7 for $f(x) = x^n$. Finally, the inequality (3.33) is immediate by (3.27) and (3.32). ■

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KHALED MEHREZ. FACULTÉ DE SCIENCES DE TUNIS, UNIVERSITÉ TUNIS-EL MANAR, TUNISIA.
E-mail address: k.mehrez@yahoo.fr

PRAVEEN AGARWAL. DEPARTMENT OF MATHEMATICS, ANAND INTERNATIONAL COLLEGE OF ENGINEERING, JAIPUR, RAJASTHAN,
INDIA.

INTERNATIONAL CENTRE FOR BASIC AND APPLIED SCIENCE, INDIA.
E-mail address: goyal.praveen@gmail.com